

The Initial Boundary Value Problems for Hyperbolic Conservation Laws with Relaxation

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The hyperbolic conservation laws with relaxation appear in many physical models such as those for gas dynamics with thermo-non-equilibrium, elasticity with

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relating the boundary data and far field states are found for the existence of the boundary layers. Also, it is proven that the boundary layers thus obtained are non-linearly stable. © 1996 Academic Press, Inc.

1. INTRODUCTION

The phenomenon of relaxation is important in many physical situations, such as the kinetic theory of nonatomic gases, the elasticity with memory, the gas flow with thermo-non-equilibrium, and the traffic flow, etc. We are interested in the phenomena of relaxation, particularly the question of stability and long time effects of relaxation on the boundary layers.

Consider the following system of two quasilinear hyperbolic equations with a boundary $x = \sigma t$ where σ is a constant boundary speed:

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{\partial f(u, v)}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial g(u, v)}{\partial x} &= h(u, v)\end{aligned}\quad 0 \leq t, \quad \sigma t \leq x < \infty. \quad (1.1)$$

The first equation is the conservation law and the second is the rate equation. We define boundary layers, as the solutions of the system (1.1), which is propagating at the same speed σ as the boundary. The main purpose of this paper is to study the existence and the stability of the boundary layers.

The system (1.1) is assumed to be strictly hyperbolic with the characteristics: $\lambda_1(u, v) < \lambda_2(u, v)$.

$$\begin{aligned} \frac{\partial(f, g)}{\partial(u, v)} r_i &= \lambda_i r_i \quad i = 1, 2 \\ \lambda_1 &= \frac{1}{2} \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} - \sqrt{\left(\frac{\partial f}{\partial u} - \frac{\partial g}{\partial v} \right)^2 + 4 \frac{\partial f}{\partial v} \frac{\partial g}{\partial v}} \\ \lambda_2 &= \frac{1}{2} \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} + \sqrt{\left(\frac{\partial f}{\partial u} - \frac{\partial g}{\partial v} \right)^2 + 4 \frac{\partial f}{\partial v} \frac{\partial g}{\partial v}}. \end{aligned} \quad (1.2)$$

Often $h(u, v)$ takes the following form:

$$h(u, v) = \frac{v_*(u) - v}{\tau(u)}$$

for a equilibrium function $v_*(u)$ and a positive function $\tau(u)$, which is the relaxation time. We make a more general assumption as follows:

$$\frac{\partial h(u, v)}{\partial v} < 0, \quad h(u, v_*(u)) = 0 \quad (1.3)$$

for all (u, v) under consideration. The equilibrium set

$$\Gamma \stackrel{\text{def}}{=} \{(u, v); v = v_*(u)\}$$

is a curve in (u, v) plane.

When the solution is close to equilibrium, one may approximate the rate equation with $v = v_*(u)$ and replace the conservation law by the following equilibrium equation:

$$\frac{\partial u}{\partial t} + \frac{\partial f_*(u)}{\partial x} = 0, \quad f_*(u) \stackrel{\text{def}}{=} f(u, v_*(u)). \quad (1.4)$$

The equilibrium characteristic for (1.3) is

$$\lambda_* = f'_*(u) = f'_u(u, v_*(u)) + f'_v(u, v_*(u)) v'_*(u). \quad (1.5)$$

We also assume the subcharacteristic condition:

$$\lambda_1 < \sigma < \lambda_2 \quad \text{for all } (u, v) \text{ under consideration.} \quad (1.6)$$

which is needed for well-posedness of the initial and boundary problems. Furthermore we assume that the conservation law and the rate equation are strongly coupled:

$$\frac{\partial f}{\partial v}(u, v) \neq 0 \quad \text{for all } (u, v) \text{ under consideration,}$$

We may assume without loss of generality that

$$\frac{\partial f}{\partial v}(u, v) > 0 \quad \text{for all } (u, v) \text{ under consideration.} \quad (1.7)$$

Liu [6] studied the Cauchy problem for (1.1) and gave the correspondence between the admissibility condition of shock waves of (1.4) and the existence of traveling waves of (1.1). Also, the stability of traveling waves is proved in [6]. The question of zero relaxation is discussed in Chen and Liu [1], for the specific physical model. For more general cases, the reader should refer to Chen, Levermore and Liu [2].

In this paper, we investigate the initial boundary value problems. In the next chapter, we give the sufficient and necessary condition to assure the existence of boundary layers. Also, we discuss the relation between this condition and the admissibility condition in [6]. In the last chapter, we prove the stability of these boundary layers by an a-priori estimate. For this, we use the energy method. The subcharacteristic condition (1.6) allows us to partially fix the boundary condition constant. Then on boundary, the differential equations (1.1) is simplified for the evaluations of boundary terms in the energy estimates. The assumption $\sigma > (\lambda_1 + \lambda_2)/2$ on the boundary plays a crucial role in these evaluations.

2. EXISTENCE OF A BOUNDARY LAYER

2.1. *Sufficient and Necessary Conditions*

In this section, we give the sufficient and necessary conditions for the existence of the boundary layer wave for (1.1).

Suppose that $(u, v)(x, t) = (\varphi, \psi)(x - \sigma t)$ is a boundary layer for (1.1). A boundary layer is a permanent wave connecting the states $(u_+, v_+) = (\varphi, \psi)(\infty)$ and $(u_0, v_0) = (\varphi, \psi)(0)$. Clearly the point (u_+, v_+) should be a permanent solution for (1.1), therefore it is an equilibrium state:

$$v_+ = v_*(u_+). \quad (2.1)$$

Substituting $(u, v)(x, t) = (\varphi, \psi)(x - \sigma t)$ in (1.1), we have

$$\begin{aligned} -\sigma\varphi'(\xi) + f(\varphi(\xi), \psi(\xi))' &= 0 \\ -\sigma\psi'(\xi) + g(\varphi(\xi), \psi(\xi))' &= h(\varphi(\xi), \psi(\xi)) \end{aligned} \quad (2.2)$$

where $\xi = x - \sigma t$. Integrating the first equation of (2.2) over $0 \leq \xi < +\infty$, we obtain

$$-\sigma \int_0^\infty \varphi(\xi)' d\xi + \int_0^\infty f(\varphi(\xi), \psi(\xi))' d\xi = 0,$$

namely,

$$-\sigma[\varphi(\infty) - \varphi(0)] + [f(\varphi(\infty), \psi(\infty)) - f(\varphi(0), \psi(0))] = 0.$$

Using (2.1), we obtain

$$-\sigma(u_+ - u_0) + f(u_+, v_*(u_+)) - f(u_0, v_0) = 0. \quad (2.3)$$

The equation (2.3) corresponds to the Rankine–Hugoniot condition for shock waves for the initial value problems. Integrating the first equation of (2.2) over $[0, \xi]$ (where ξ is an arbitrary positive constant), we obtain:

$$\begin{aligned} \sigma\varphi(\xi) + f(\varphi(\xi), \psi(\xi)) &= -\sigma u_0 + f(u_0, v_0) \\ &= -\sigma u_+ + f(u_+, v_+). \end{aligned} \quad (2.4)$$

Hence, we introduce the curve L in (u, v) plane:

$$L \stackrel{\text{def}}{=} \left\{ (u, v) \in \mathbf{R}^2; \begin{aligned} &-\sigma u + f(u, v) = -\sigma u_0 + f(u_0, v_0) = -\sigma u_+ + f(u_+, v_+) \\ &u \text{ is between } u_0 \text{ and } u_+ \end{aligned} \right\}.$$

When (u_0, v_0) and (u_+, v_+) are connected by the boundary layer, we have:

$$L = \{(\varphi, \psi)(\xi); \xi \in \mathbf{R}_+\}.$$

Note when $u_0 = u_+$, the only solution allowed is constant.

Along the boundary layer $(\varphi(\xi), \psi(\xi))$, we have from (1.4) and (2.2) that, if $(\lambda_1 - \sigma)(\lambda_2 - \sigma) \neq 0$,

$$\varphi' = \frac{-f_v h}{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}, \quad (2.5)$$

$$\psi' = \frac{(f_u - \sigma) h}{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}, \quad (2.6)$$

$$h' = \frac{(\mu - \sigma) h_v h}{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}, \quad (2.7)$$

$$(\psi - v_*(\varphi))' = \frac{(f_u - \sigma) h}{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}, \quad (2.8)$$

where μ is a dynamic subcharacteristic, defined as

$$\mu \stackrel{\text{def}}{=} \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \frac{\partial h}{\partial u} \left(\frac{\partial h}{\partial v} \right)^{-1}. \quad (2.9)$$

Physically, μ is a function which governs propagations of disturbances created by various perturbations.

For given fixed $(u_+, v_*(u_+))$ and σ , we have the following lemma.

LEMMA 2.1. *If a boundary layer connects $(u_+, v_*(u_+) = v_+)$ and (u_0, v_0) , then (u_0, v_0) is not an equilibrium state, i.e., $(u_0, v_0) \notin \Gamma$.*

Proof. This can be proved by contradiction. Assume $(u_0, v_0) \in \Gamma$. From (1.2) and (2.7), we have $h(0) = h'(0) = 0$. By the Hadamard's Lemma, there exists some continuous function $g(\xi)$ satisfying $h(\xi) = \xi g(\xi)$. Substituting this in (2.5), we have

$$\varphi' = \frac{-f_v \xi g(\xi)}{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}.$$

Since $L = \{(\varphi, \psi)(\xi); 0 \leq \xi < \infty\}$ is a compact set in \mathbf{R}^2 , whose end points are (u_0, v_0) and (u_+, v_+) , $(\lambda_1 - \sigma)(\lambda_2 - \sigma)$ is uniformly negative i.e., there exists a certain constant $C < 0$ s.t. $(\lambda_1 - \sigma)(\lambda_2 - \sigma) < C$ for arbitrary ξ . Because $\varphi' \neq 0$, we have

$$\begin{aligned} d\xi &= -\frac{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}{f_v \xi g(\xi)} d\varphi, \\ \xi_0 &= -\int_0^{\xi_0} \frac{(\lambda_1 - \sigma)(\lambda_2 - \sigma)}{f_v \xi g(\xi)} d\varphi \quad \text{for any } \xi_0. \end{aligned}$$

Since the last integral diverges, we get a contradiction. \blacksquare

Lemma 2.1 tells us that $L = \{(\varphi, \psi)(\xi)\}$ never intersects Γ except at (u_+, v_+) .

Next we look for the necessary condition for the existence of the boundary layer connecting (u_0, v_0) and (u_+, v_+) where $v_+ = v_*(u_+)$, $v_0 \neq v_*(u_0)$, assuming the existence of the boundary layer $(\varphi, \psi)(x - \sigma t)$ connecting

them. Because of the Lemma 2.1, if $h > 0$, then we have $h > 0$ on L except at (u_+, v_+) . Hence, we have the following relationships:

$$\begin{aligned} (u_0, v_0) \in \mathbf{A} \cup \mathbf{B} &\Leftrightarrow h < 0 && \text{on } L \setminus \{(u_+, v_+)\} \\ (u_0, v_0) \in \mathbf{C} \cup \mathbf{D} &\Leftrightarrow h > 0 && \text{on } L \setminus \{(u_+, v_+)\} \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &\stackrel{\text{def}}{=} \{(u, v); h < 0, u_0 < u_+\} \\ \mathbf{B} &\stackrel{\text{def}}{=} \{(u, v); h < 0, u_0 > u_+\} \\ \mathbf{C} &\stackrel{\text{def}}{=} \{(u, v); h > 0, u_0 < u_+\} \\ \mathbf{D} &\stackrel{\text{def}}{=} \{(u, v); h > 0, u_0 > u_+\}. \end{aligned}$$

From (2.3), when (u_+, v_+) and u_0 are fixed, σ can be viewed as a function of v_0 . Differentiating (2.3) with respect to v_0 , we obtain

$$\sigma'(v_0) = \frac{f_v(u_0, v_0)}{u_0 - u_+}. \quad (2.10)$$

By (1.7), we have

$$u_0 < u_+ \Leftrightarrow \sigma'(v_0) < 0, \quad \text{and} \quad (2.11)$$

$$u_0 > u_+ \Leftrightarrow \sigma'(v_0) > 0. \quad (2.12)$$

Consequently, we obtain the following table (cf. Fig. 2.1).

$$\begin{aligned} [1] \quad u_0 < u_+ &\Leftrightarrow \sigma'(v_0) < 0 \\ & (u_0, v_0) \in \mathbf{A} \Leftrightarrow v_0 > v_*(u_0) \Leftrightarrow \sigma(v_0) < \sigma(v_*(u_0)) \\ & (u_0, v_0) \in \mathbf{C} \Leftrightarrow v_0 < v_*(u_0) \Leftrightarrow \sigma(v_0) > \sigma(v_*(u_0)) \\ [2] \quad u_0 > u_+ &\Leftrightarrow \sigma'(v_0) > 0 \\ & (u_0, v_0) \in \mathbf{B} \Leftrightarrow v_0 > v_*(u_0) \Leftrightarrow \sigma(v_0) > \sigma(v_*(u_0)) \\ & (u_0, v_0) \in \mathbf{D} \Leftrightarrow v_0 < v_*(u_0) \Leftrightarrow \sigma(v_0) < \sigma(v_*(u_0)). \end{aligned}$$

With this relationships in one's mind, we make the following observation.

LEMMA 2.2. *If there exists a boundary layer $(\varphi, \psi)(x - \sigma t)$, then $(u_0, v_0) \in \mathbf{B} \cup \mathbf{C}$. Hence, $L \subseteq \mathbf{B} \cup \mathbf{C} \cup \{(u_+, v_+)\}$.*

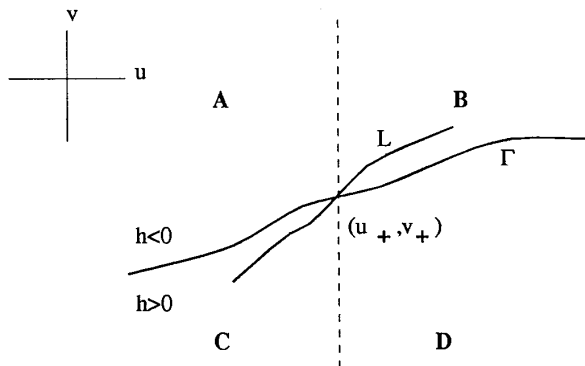


FIGURE 2.1

Proof. By (1.6), (1.7) and (2.5), we have that φ' and h have the same sign. Since L intersects Γ at only (u_+, v_+) , the end point of L , if $h > 0$ ($h < 0$) at (u_0, v_0) , then $h > 0$ ($h < 0$) on L . Hence, if φ is increasing (decreasing) at (u_0, v_0) , then φ is increasing (decreasing) on L .

If $(u_0, v_0) \in \mathbf{A}$, then $u_0 = \varphi(0) < u_+ = \varphi(\infty)$. But $h(u_0, v_0) < 0$ implies $\varphi' < 0$ on L . This gives a rise to a contradiction. In a similar manner, a contradiction follows from the condition $(u_0, v_0) \in \mathbf{D}$. ■

THEOREM 2.3. Suppose $\lambda_1 < \sigma < \lambda_2$ (1.6) and $f_v > 0$ (1.7) on L . There exists a boundary layer connecting (u_0, v_0) and (u_+, v_+) if and only if $(u_0, v_0) \in \mathbf{B} \cup \mathbf{C}$ and h never assumes 0 on $L \setminus \{(u_+, v_+)\}$.

Proof. (\Rightarrow) The proof follows from Lemma 2.1 and Lemma 2.2.

(\Leftarrow) (2.5) and (2.6) can be solved from (u_0, v_0) until the critical point (u_+, v_+) by the standard O.D.E. theory. Thus we have the proof. ■

The following theorem and its corollary are weaker than the Theorem 2.3, i.e., it does not give the necessary condition. However they will prove to be more applicable in our context later.

THEOREM 2.4. Suppose $\lambda_1 < \sigma < \lambda_2$ (1.6) and $f_v > 0$ (1.7) on L . When $\lambda_*(u) < \sigma$ on L , there exists boundary layer connecting (u_0, v_0) and (u_+, v_+) . Also when $\lambda_*(u) > \sigma$ on L , there does not exist boundary layer connecting (u_0, v_0) and (u_+, v_+) .

COROLLARY 2.5. If $f_*(u)$ is convex, $u_+ < u_0$ and $\lambda_*(u_0) < \sigma$, then there exists a boundary layer.

Before giving the proof of Theorem 2.4, we need the following Lemma 2.6.

LEMMA 2.6. If $\lambda_*(u) < \sigma$ on L , then $(u_0, v_0) \in \mathbf{B} \cup \mathbf{C}$

Proof. We will prove the case $u_0 < u_+$; the case $u_0 > u_+$ can be proved analogously. Assume (u_0, v_0) lies in \mathbf{A} or its closure, then we have $v_0 \geq v_*(u_0)$. Since $(u_0, v_0) \in \mathbf{A}$, we obtain: $\sigma(u_0 - u_+) = f(u_0, v_0) - f(u_+, v_+)$. As $f_v > 0$, we have $f(u_0, v_*(u_0)) \leq f(u_0, v_0)$. Therefore,

$$\sigma(v_0 - v_*(u_0)) \geq f(u_0, v_*(u_0)) - f(u_+, v_+) = f_*(u_0) - f_*(u_+). \quad (2.13)$$

By the mean value theorem, there exists a certain number $\hat{u} \in (u_0, u_+)$ such that

$$f'_*(\hat{u})(u_0 - u_+) = f_*(u_0) - f_*(u_+). \quad (2.14)$$

From (2.13) and (2.14), we have $\sigma(u_0 - u_+) \geq \lambda_*(\hat{u})(u_0 - u_+)$. Since $u_0 - u_+ < 0$, $\sigma \leq \lambda_*(\hat{u})$. This contradicts to the assumption. ■

Proof of Theorem 2.4. Suppose $\lambda_*(u) < \sigma$. By Lemma 2.2, we have $(u_0, v_0) \in \mathbf{B} \cup \mathbf{C}$. Since the case $(u_0, v_0) \in \mathbf{B}$ can be dealt analogously, We give the proof for the case $(u_0, v_0) \in \mathbf{C}$ by contradiction. Since $\Gamma = \{h = 0\}$ is continuous curve, it is sufficient to show that L and Γ intersect with each other only at (u_+, v_+) . Assume L and Γ intersect at the point (\tilde{u}, \tilde{v}) ($\neq (u_+, v_+)$). By the definition of L , we have: $\sigma(\tilde{u} - u_+) = f(\tilde{u}, \tilde{v}) - f(u_+, v_+)$. Since (\tilde{u}, \tilde{v}) and (u_+, v_+) are included in Γ , (i.e., $\tilde{v} = v_*(\tilde{u})$, $v_+ = v_*(u_+)$), we have $\sigma(\tilde{u} - u_+) = f_*(\tilde{u}) - f_*(u_+)$. By the mean value theorem, there exists $\hat{u} \in (\tilde{u}, u_+)$ such that $f_*(\tilde{u}) - f_*(u_+) = \lambda_*(\hat{u})(\tilde{u} - u_+)$. Hence

$$\sigma(\tilde{u} - u_+) = \lambda_*(\hat{u})(\tilde{u} - u_+), \quad \text{thus} \quad \sigma = \lambda_*(\hat{u}).$$

By the continuity of the curve L , there is some \hat{v} such that $(\hat{u}, \hat{v}) \in L$. This contradicts to the condition $\lambda_*(u) < \sigma$ on L .

Next, suppose $\lambda_*(u) > \sigma$. We give the proof for the case $u_0 < u_+$, only. Assume the boundary layer $(\varphi, \psi)(x - \sigma t)$ exists. From Theorem 2.3, we have $L \subseteq \mathbf{C} \cup \{(u_+, v_+)\}$. By the definition of L , we have

$$\sigma(u_0 - u_+) = f(u_0, v_0) - f(u_+, v_+).$$

Since $f_v > 0$, it follows that $f(u_0, v_0) < f(u_0, v_*(u_0))$, which implies

$$\sigma(u_0 - u_+) < f(u_0, v_*(u_0)) - f(u_+, v_+) = f_*(u_0) - f_*(u_+).$$

By the mean value theorem, there exists $\tilde{u} \in (u_0, u_+)$ such that

$$f'_*(\tilde{u})(u_0 - u_+) = f_*(u_0) - f_*(u_+).$$

Consequently, $\sigma > \lambda_*(\tilde{u})$. Because of the continuity of L , we get a contradiction. ■

2.2. Admissibility of Boundary Layers

Here we consider the relationship between sufficient conditions discussed in the last section and an admissibility condition in [6]. We refer to the following theorem from [6].

THEOREM 2.7. *If*

$$\sigma < \frac{f(\tilde{u}, v_*(\tilde{u})) - f(u_-, v_-)}{\tilde{u} - u_-} \quad (2.15)$$

on $\tilde{L} = \{(\tilde{u}, \tilde{v}); \sigma\tilde{u} - f(\tilde{u}, \tilde{v}) = \sigma u_{\pm} - f(u_{\pm}, v_{\pm}) \text{ } u \text{ is between } u_- \text{ and } u_+\}$, then (u_-, v_-) and (u_+, v_+) can be connected by traveling wave with speed σ .

As easily seen, when $(u_0, v_0) \in \tilde{L}$, there exists a boundary layer (assume σ is fixed) connecting (u_0, v_0) and (u_+, v_+) , if (u_-, v_-) and (u_+, v_+) can be connected. Hence, the admissibility above is one of the sufficient conditions for existence of a traveling wave of our boundary problems. The following lemma gives an insight to the relevance of the admissibility condition in our context.

LEMMA 2.8. *The admissibility condition (2.15) is equivalent to the condition that \tilde{L} lies above (below) Γ when $\tilde{u} < u_+$ ($\tilde{u} > u_+$), and it only intersects with Γ at its end points, (u_-, v_-) and (u_+, v_+) .*

Proof. (\Rightarrow) We prove the case $u_- < u_+$ only. For arbitrary point (\tilde{u}, \tilde{v}) on \tilde{L} , we have

$$\sigma = \frac{f(u_+, v_+) - f(u_-, v_-)}{u_+ - u_-} < \frac{f(\tilde{u}, v_*(\tilde{u})) - f(u_-, v_*(u_-))}{\tilde{u} - u_-}.$$

Since $\tilde{u} > u_-$, it follows that

$$\sigma\tilde{u} - f(\tilde{u}, v_*(\tilde{u})) < \sigma u_- - f(u_-, v_*(u_-)) = \sigma\tilde{u} - f(\tilde{u}, \tilde{v}).$$

Hence, $f(\tilde{u}, \tilde{v}) < f(\tilde{u}, v_*(\tilde{u}))$. Since $f_v > 0$, we have $\tilde{v} < v_*(\tilde{u})$. So (\tilde{u}, \tilde{v}) remains under the curve Γ .

(\Leftarrow) We prove the case $u_- < u_+$ only. As \tilde{L} lies under the curve Γ , for arbitrary point (\tilde{u}, \tilde{v}) on \tilde{L} , we have $\tilde{v} < v_*(\tilde{u})$. By the same discussion as above, we obtain

$$\sigma < \frac{f(\tilde{u}, v_*(\tilde{u})) - f(u_-, v_*(u_-))}{\tilde{u} - u_-}. \quad \blacksquare$$

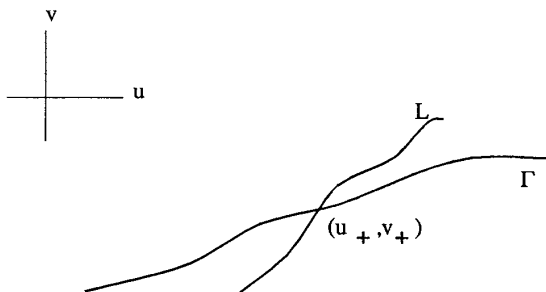


FIGURE 2.2

2.3. Local Problem

In this section we consider the case that the point (u_+, v_+) (recall $v_*(u_+) = v_+$) and the boundary speed σ is given. We define the curve \hat{L} as follows

$$\hat{L} = \{(u, v) \in \mathbf{R}^2; -\sigma u + f(u, v) = -\sigma u_+ + f(u_+, v_+)\}.$$

We have the following theorem.

THEOREM 2.9. *Assume (1.3), (1.6) and (1.7) hold.*

(1) *If $\lambda_*(u_+) < \sigma$, then we can find (u_0, v_0) on \hat{L} such that (u_0, v_0) and (u_+, v_+) can be connected by a boundary layer. Here, (u_0, v_0) can be found in the both sides of (u_+, v_+)*

(2) *If $\lambda_*(u_+) = \sigma$ and $\lambda'_*(u_+) \neq 0$, then we can find (u_0, v_0) on \hat{L} such that (u_0, v_0) and (u_+, v_+) can be connected by boundary layer. When f_* is convex (i.e., $\lambda'_*(u_+) > 0$), we have $u_0 < u_+$. When f_* is concave (i.e., $\lambda'_*(u_+) < 0$), we have $u_0 > u_+$.*

(3) *If $\lambda_*(u_+) > \sigma$, there does not exist any boundary layers.*

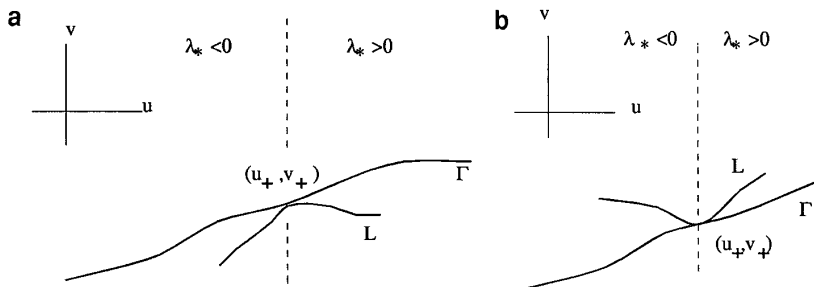


FIGURE 2.3

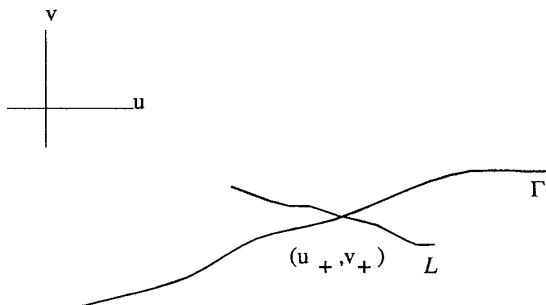


FIGURE 2.4

Proof. Proof is a direct consequence of the following Lemma and Theorem 2.3. ■

LEMMA 2.10. (1) If $\lambda_*(u_+) < \sigma$, then \hat{L} intersects with Γ transversally upward at (u_+, v_+) , (see Fig. 2.2).

(2) If $\lambda_*(u_+) = \sigma$ and $\lambda'_*(u_+) \neq 0$, then \hat{L} intersects with Γ tangentially at (u_+, v_+) , (see Fig. 2.3).

(3) If $\lambda_*(u_+) > \sigma$, then \hat{L} intersects with Γ transversally downward at (u_+, v_+) , (see Fig. 2.4).

Proof. One of tangent vectors of \hat{L} is given by $V = (f_v, \sigma - f_u)$. Thus the directional differentiation of $v - v_*(u)$ along with \hat{L} (taking the direction of u as positive) is given by;

$$\begin{aligned} d_L(v - v_*(u)) &= \nabla(v - v_*(u)) \frac{V}{|V|} \\ &= \frac{1}{|V|} (-v'_*(u), 1)(f_v, \sigma - f_u) \\ &= \frac{1}{|V|} \{ \sigma - (v'_*(u) f_v + f_u) \}. \end{aligned}$$

Hence, we have

$$d_L(v - v_*(u))|_{(u_+, v_+)} = \frac{1}{|V|} (\sigma - \lambda_*(u_+)). \quad (2.16)$$

This proves the lemma. ■

At this stage, for a given (u_+, v_+) , we would like to know conditions for (u_0, v_0) which guarantees the existence of a boundary layer connecting (u_+, v_+) and (u_0, v_0) . The following theorem gives those conditions.

THEOREM 2.11. *Suppose that*

$$f_* \text{ is convex,} \quad (2.18)$$

and also suppose (1.3), (1.6) and (1.7) hold.

Assume further that for given (u_+, v_+) , (u_0, v_0) satisfies one of the following conditions;

- (a) $u_0 > u_+$ and $v_0 > v_*(u_0)$
- (b) $u_0 < u_+$ and $\lambda_*(u_+) u_0 - f(u_0, v_0) \geq -\lambda_*(u_+) u_+ - f(u_+, v_+)$.

Then, there exists some σ such that the boundary condition (u_0, v_0) and the infinite state (u_+, v_+) at $(+\infty, t)$ can be connected by a boundary layer with the speed $\sigma (\geq \lambda_(u_+))$.*

Proof. The proof follows in a straightforward way from the continuity of \hat{L} and the previous discussions. ■

Remark 2.1. In act, it is true that if such a boundary layer exists, either the condition (a) or (b) has to hold.

Remark 2.2. $\sigma = \lambda_*(u_+)$ never holds for the condition (a).

Until now, we have assumed (1.7). But as easily seen, the discussion above is also valid for the case $f_v < 0$. In next section, we investigate the stability of boundary layer under the condition $f_v \neq 0$ instead of $f_v > 0$.

3. STABILITY OF BOUNDARY LAYERS

3.1. Initial and Boundary Conditions

Let $(\varphi, \psi)(x - \sigma t)$ be a boundary layers satisfying (1.1) with boundary speed σ and also satisfying $\lambda_1 < \sigma < \lambda_2$ (1.6). Let $(u, v)(x, t)$ be a solution of (1.1) whose initial data is a perturbation of (φ, ψ) ;

$$\begin{aligned} (u, v)(x, 0) &= (\varphi, \psi)(x) + (\bar{u}, \bar{v})(x, 0) \\ (\bar{u}, \bar{v})(\infty, 0) &= 0 \\ \bar{u}(0, 0) &= 0. \end{aligned} \quad (3.1)$$

Our purpose is to show that the solution (u, v) exists globally in time and tends to $(\varphi, \psi)(x - \sigma t)$ as $t \rightarrow \infty$ when perturbation $(\bar{u}, \bar{v})(x, 0)$ is sufficiently small in some sense. The basic hyperbolic P.D.E. theory tells us this problem needs the boundary condition because subcharacteristic

condition $\lambda_1(u, v) < \sigma < \lambda_2(u, v)$ holds when perturbation $(\bar{u}, \bar{v})(x, 0)$ is small. Thus, we will look for the solution (u, v) satisfying;

$$u(\sigma t, t) = \varphi(0) = u_0 \quad (3.2)$$

We assume that the compatibility condition holds at $(0, 0)$ with respect to the 1st derivatives. It implies the conservation law in (1.1) holds at $(0, 0)$, namely

$$\frac{\partial f(u, v)}{\partial x} = 0.$$

By changing variables $x - \sigma t \rightarrow \xi$ and $t \rightarrow T$, (1.1) becomes;

$$\begin{aligned} \frac{\partial u}{\partial T} + \frac{\partial \bar{f}}{\partial \xi}(u, v) &= 0 \\ \frac{\partial v}{\partial T} + \frac{\partial \bar{g}}{\partial \xi}(u, v) &= h(u, v). \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \bar{f}(u, v) &= f(u, v) - \sigma u \\ \bar{g}(u, v) &= g(u, v) - \sigma u. \end{aligned}$$

Since (3.3) is also hyperbolic and $(\partial \bar{f} / \partial v)(u, v) = \partial f / \partial v \neq 0$, we may assume

$$\sigma = 0 \quad (3.4)$$

without loss of generality. We will use (x, t) instead of (ξ, T) and suppose $\sigma = 0$ from now on. Furthermore, we make the following assumptions:

Suppose that (u_-, v_-) and (u_+, v_+) are connected by a traveling wave satisfying the admissibility condition. Furthermore, $(u_0, v_0) = (\varphi, \psi)(0)$ is a certain point on the curve \tilde{L} . This condition implies $|\mu - \sigma| \ll 1$ when traveling wave is weak (i.e., $|u_- - u_+| \ll 1$). We also suppose f_* is convex. By the Theorem 2.11 and the Remark 2, we have $\lambda_*(u_+) < \sigma$. Define \hat{u} to be a number satisfying $\lambda_*(\hat{u}) = 0$. Note, here, that by sending $|u_- - u_+|$ to zero, the curve \tilde{L} degenerates to a point $(\hat{u}, v_*(\hat{u}))$. The following discussion is valid under the condition (u_+, v_+) , (u_0, v_0) and $(\hat{u}, v_*(\hat{u}))$ are sufficiently near to each other.

Here, we introduce some notations, taken from [6].

$$\begin{aligned} Z(x, t) &\stackrel{\text{def}}{=} - \int_x^\infty u(y, t) - \varphi(y) dy \\ w(x, t) &\stackrel{\text{def}}{=} v(x, t) - \psi(x). \end{aligned} \quad (3.5)$$

It can be immediately verified from (3.1) and (3.2) that:

$$\begin{aligned}
 Z_x(x, t) &= \varphi(x) - u(x, t) \\
 Z_x(0, t) &= \varphi(0) - u(0, t) = 0 \\
 Z_{xt}(0, t) &= 0 \\
 Z_{xtt}(0, t) &= 0.
 \end{aligned} \tag{3.6}$$

Also, define

$$\begin{aligned}
 \delta(\tau) \stackrel{\text{def}}{=} & |u_+ - u_-| + |u_+ - u_0| + \max_{0 \leq t \leq \tau} \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, t)|^2 \\
 & + \sum_{j=0}^2 |\nabla^j w(x, t)|^2 dx.
 \end{aligned} \tag{3.7}$$

3.2. Main Lemma

In order to show the existence of a global solution in terms of t for the system (1.1), (3.1) and (3.2), we need the following a-priori estimate. This main Lemma is proved by energy estimate in the next two sections to come.

LEMMA 3.1. *If $\delta(\tau)$ is sufficiently small, then we have*

$$\begin{aligned}
 & Z(0, \tau)^2 + \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, \tau)|^2 + \sum_{j=0}^2 |\nabla^j w(x, \tau)|^2 dx \\
 & + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + \sum_{j=0}^3 |\nabla^j Z|^2 + \sum_{j=0}^2 |\nabla^j w(x, \tau)|^2 dx dt \\
 & \leq O(1) \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 + \sum_{j=0}^2 |\nabla^j w(x, 0)|^2 dx.
 \end{aligned} \tag{3.8}$$

Recall that both (u, v) and (φ, ψ) satisfy (1.1). Take the difference of the two equations of the conservation law in (1.1), satisfied by (u, v) and (φ, ψ) respectively. Integrate it with respect to x , and obtain:

$$Z_t + f(\varphi + Z_x, \psi + w) - f(\varphi, \psi) = 0. \tag{3.9}$$

Now, take the difference of the two rate equations in (1.1), satisfied by (u, v) and (φ, ψ) respectively, and obtain:

$$w_t + [g(\varphi + Z_x, \psi + w) - g(\varphi, \psi)]_x = h(\varphi + Z_x, \psi + w) - h(\varphi, \psi) \tag{3.10}$$

Since $f_v \neq 0$, we can w in terms of Z :

$$w = f_v^{-1}(-Z_t - f_u Z_x - Q_0(f)), \quad (3.11)$$

where

$$\begin{aligned} Q_0(f) &\stackrel{\text{def}}{=} f(Z_x + \varphi, w + \psi) - f(\varphi, \psi) - f_u Z_x - f_v w \\ &= O(1)(Z_x^2 + w^2) \end{aligned} \quad (3.12)$$

$$f_u = f_u(\varphi, \psi) \quad \text{and} \quad f_v = f_v(\varphi, \psi).$$

The second equality of $Q_0(f)$ above is valid under the condition $\max_{0 \leq \tau \leq t} \delta(t) \ll 1$ (more precisely, $\max_{0 \leq \tau \leq t} \delta(t)$ is bounded). Furthermore, for small $Z_x^2 + w^2$, we have

$$w = W(Z_t, Z_x, \varphi, \psi) = O(1)(|Z_t| + |Z_x|). \quad (3.13)$$

Substituting in (3.11) into $-f_v$ (3.10), we have

$$\begin{aligned} &Z_{tt} + (\lambda_1 + \lambda_2) Z_{xt} + \lambda_1 \lambda_2 Z_{xx} - h_v(Z_t + \mu Z_x) \\ &= (f_v^{-1} g_v f_{vx} - g_{vx})(Z_t + f_u Z_x) + Q_0(f)(f_v^{-1} g_v f_{vx} - g_{vx} + h_v +) \\ &\quad - f_v Q_0(h) - Q_0(f)_t - g_v Q_0(f)_x + f_v Q_0(g)_x + (f_v g_{ux} - g_v f_{ux}) Z_x \\ &\stackrel{\text{def}}{=} R(x, t). \end{aligned} \quad (3.14)$$

All known functions $\lambda_1, \lambda_2, f_u, f_v, h_v$ and μ are functions of x only, independent of t .

We conclude from integrating (3.14) $Z, (3.14) Z_t, (3.14)_x Z_x, (3.14)_t Z_{tt}, (3.14)_x Z_{tx}, (3.14)_{xx} Z_{xx}, (3.14)_{xx} Z_{xxt}, (3.14)_{tx} Z_{txx}$ and $(3.14)_{tt} Z_{ttt}$, over $0 \leq x < \infty, 0 \leq t < \tau$, respectively, that

$$\begin{aligned} &\int_0^\infty Z^2(x, \tau) dx + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + Z_x^2 dx dt \\ &\leq C \int_0^\infty Z^2(x, 0) + Z_t^2(x, 0) + Z_t^2(x, \tau) dx + \varepsilon \int_0^\infty Z_x^2(x, \tau) dx \\ &\quad + C \int_0^\tau \int_0^\infty Z_t^2 dx dt + C \int_0^\tau (\lambda_1 + \lambda_2)(0) Z_t Z(0, t) \\ &\quad - \frac{1}{2} |(\mu h_v)(0)| Z^2(0, t) dt + \varepsilon \int_0^\tau Z_t^2(x, 0) dt \end{aligned} \quad (3.15)$$

$$\begin{aligned}
& \int_0^\infty Z_t^2(x, \tau) + Z_x^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_t^2 dx dt \\
& \leq C \int_0^\infty Z_t^2(x, 0) + Z_x^2(x, 0) dx \\
& \quad + \varepsilon \int_0^\tau \int_0^\infty Z_{xx}^2 + Z_{tt}^2 + Z_{xt}^2 + Z_x^2 dx dt \\
& \quad + C \int_0^\tau \left(\frac{\lambda_1 + \lambda_2}{2}(0) + \varepsilon \right) Z_t^2(0, t) dt
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& \int_0^\infty Z_x^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{xx}^2 dx dt \\
& \leq C \int_0^\infty Z_x^2(x, 0) dx + C \int_0^\tau \int_0^\infty Z_{tt}^2 + Z_{xt}^2 dx dt \\
& \quad + \varepsilon \int_0^\tau \int_0^\infty Z_{xxx}^2 + Z_{xxt}^2 + Z_{xtt}^2 + Z_t^2 + Z_x^2 dx dt
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& \int_0^\infty Z_{tt}^2(x, \tau) + Z_{xt}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{tt}^2 dx dt \\
& \leq C \int_0^\infty Z_{tt}^2(x, 0) + Z_{xt}^2(x, 0) dx \\
& \quad + \varepsilon \int_0^\tau \int_0^\infty Z_{xtt}^2 + Z_{ttt}^2 + Z_{xxt}^2 + Z_{xt}^2 + Z_{xx}^2 + |\nabla Z|^2 dx dt \\
& \quad + C \int_0^\tau \left(\frac{\lambda_1 + \lambda_2}{2}(0) \right) Z_{tt}^2(x, 0) dt
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
& \int_0^\infty Z_{tx}^2(x, \tau) + Z_{xx}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{xt}^2 dx dt \\
& \leq C \int_0^\infty Z_{tx}^2(x, 0) + Z_{xx}^2(x, 0) dx \\
& \quad + \varepsilon \int_0^\tau \int_0^\infty Z_{xtt}^2 + Z_{xxx}^2 + Z_{xxt}^2 + Z_{tt}^2 + Z_{xx}^2 + |\nabla Z|^2 dx dt
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& \int_0^\infty Z_{xx}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{xxx}^2 dx dt \\
& \leq C \int_0^\infty Z_{xx}^2(x, 0) dx + C \int_0^\infty Z_{xxx}^2(x, \tau) + Z_{xxx}^2(x, 0) dx
\end{aligned}$$

$$\begin{aligned}
& + C \int_0^\tau \int_0^\infty Z_{xtt}^2 + Z_{xxt}^2 dx dt + \varepsilon \int_0^\tau \int_0^\infty Z_{xxx}^2 + |\nabla^2 Z|^2 + |\nabla Z|^2 dx dt \\
& + C \int_0^\tau (\lambda_1 \lambda_2)(0) Z_{xxx} Z_{xx}(0, t) dt \\
& + C \int_0^\tau \int_0^\infty [-Q_0(f)_t - g_v Q_0(f)_x + f_v Q_0(g)_x]_{xx} Z_{xx} dx dt \quad (3.20)
\end{aligned}$$

$$\begin{aligned}
& \int_0^\infty Z_{txx}^2(x, \tau) + Z_{xxx}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{xxt}^2 dx dt \\
& \leq C \int_0^\infty Z_{txx}^2(x, 0) + Z_{xxx}^2(x, 0) dx \\
& + \varepsilon \int_0^\tau \int_0^\infty Z_{xxx}^2 + |\nabla^2 Z|^2 + |\nabla Z|^2 dx dt \\
& + C \int_0^\tau (\lambda_1 \lambda_2)(0) Z_{xxx} Z_{xxt}(0, t) + \frac{(\lambda_1 + \lambda_2)}{2}(0) Z_{xxt}^2(0, t) dt \\
& + C \int_0^\tau \int_0^\infty [-Q_0(f)_t - g_v Q_0(f)_x + f_v Q_0(g)_x]_{xx} Z_{xxt} dx dt \quad (3.21)
\end{aligned}$$

$$\begin{aligned}
& \int_0^\infty Z_{txx}^2(x, \tau) + Z_{xxx}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{txx}^2 dx dt \\
& \leq C \int_0^\infty Z_{txx}^2(x, 0) + Z_{xxx}^2(x, 0) dx \\
& + \varepsilon \int_0^\tau \int_0^\infty Z_{txx}^2 + |\nabla^2 Z|^2 + |\nabla Z|^2 dx dt \\
& + C \int_0^\tau \int_0^\infty [-Q_0(f)_t - g_v Q_0(f)_x + f_v Q_0(g)_x]_{tx} Z_{txx} dx dt \quad (3.22)
\end{aligned}$$

$$\begin{aligned}
& \int_0^\infty Z_{xtt}^2(x, \tau) + Z_{xxt}^2(x, \tau) dx + \int_0^\tau \int_0^\infty Z_{xtt}^2 dx dt \\
& \leq C \int_0^\infty Z_{xtt}^2(x, 0) + Z_{xxt}^2(x, 0) dx \\
& + \varepsilon \int_0^\tau \int_0^\infty Z_{xtt}^2 |\nabla^2 Z|^2 + |\nabla Z|^2 dx dt \\
& + C \int_0^\tau \frac{(\lambda_1 + \lambda_2)}{2}(0) Z_{xtt}^2(0, t) dt \\
& + C \int_0^\tau \int_0^\infty [-Q_0(f)_t - g_v Q_0(f)_x + f_v Q_0(g)_x]_{tt} Z_{xtt} dx dt \quad (3.23)
\end{aligned}$$

where C is an universal constant (independent from τ), and ε is a constant which can be made small by taking δ small.

In order to explain the principal idea in getting evaluations above, we will derive (3.15) in the next section.

3.3. Energy Estimate

For weak traveling waves (i.e., $|u_+ - u_-| \ll 1$), μ is close to λ_* , and we have (see [6]),

$$|\varphi_x| + |\psi_x| \leq C |\mu_x| \ll 1; \quad (3.24)$$

$$\mu_x < 0; \quad \text{and} \quad (3.25)$$

$$|\sigma - \mu| + |\sigma - \lambda_*| \ll 1. \quad (3.26)$$

The inequality (3.25) come from Lemma 2.8 and the assumption that is convex f_* . When $|u_+ - u_0|$ is sufficiently small, we have by (3.4),

$$\mu < 0 \quad |\mu| + |\lambda_*| \ll 1. \quad (3.27)$$

Furthermore, when we assume $|u_+ - u_0| \ll 1$, we have $\mu(0) < 0$, since $\lambda_*(u_+) < 0$.

Integrate (3.14) Z over $0 \leq t \leq \tau$ and $0 \leq x < \infty$, then using above inequalities and $h_v < 0$, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |h_v| Z^2(x, \tau) dx + \frac{1}{2} \int_0^\tau \int_0^\infty (\mu h_v)_x Z^2 dx dt + \int_0^\tau \int_0^\infty |\lambda_1 \lambda_2| Z_x^2 dx dt \\ &= \int_0^\infty \frac{1}{2} |h_v| Z^2(x, 0) + Z_t Z(x, 0) - Z_t Z(x, \tau) dx \\ &+ \int_0^\tau \int_0^\infty Z_t^2 + (\lambda_1 + \lambda_2)_x Z_t Z + (\lambda_1 + \lambda_2) Z_t Z_x + (\lambda_1 \lambda_2)_x Z_x Z dx dt \\ &+ \int_0^\tau (\lambda_1 + \lambda_2)(0) Z_t Z(0, t) - \frac{1}{2} (\mu h_v)(0) Z^2(0, t) dt \\ &+ \int_0^\tau \int_0^\infty R(x, t) Z dx dt. \end{aligned} \quad (3.28)$$

Repeating the same calculation as (3.11) for g and h , we have

$$|Q_0(f)| + |Q_0(g)| + |Q_0(h)| = O(1)(|Z_t|^2 + |Z_x|^2). \quad (3.29)$$

Using (3.24), (3.25), (3.27) and (3.29), we obtain (3.15).

(3.15) to (3.23) treat all the first three derivatives of Z . Note, however, the last integrals in (3.21) to (3.23) appear to contain the fourth derivatives

of Z . This difficulty is resolved by Taylor expansions and integration by parts. We illustrated this by considering the following terms in (3.21):

$$\begin{aligned}
& \frac{1}{2} \int_0^\tau \int_0^\infty -Q_0(f)_{tXX} Z_{tXX} dx dt \\
&= -\frac{1}{2} \int_0^\tau ([f_u] Z_{tXXX} + [f_v] w_{tXX}) Z_{tXX} + (\text{L.O.T.}) dx dt \\
&= -\frac{1}{2} \int_0^\tau \int_0^\infty ([f_u] Z_{tXXX} + [f_v](W_1 Z_{tXX} + W_2 Z_{tXXX}) Z_{tXX} \\
&\quad + (\text{L.O.T.}) dx dt
\end{aligned}$$

where

$$\begin{aligned}
[f_u] &= f_u(Z_x + \varphi, w + \psi) - f_u(\varphi, \psi) \\
&= O(1)(|Z_x| + |w|) \\
[f_v] &= f_v(Z_x + \varphi, w + \psi) - f_v(\varphi, \psi) \\
&= O(1)(|Z_x| + |w|).
\end{aligned}$$

Here, W_1 and W_2 are the first derivatives of the function W in (3.13) with respect to the first and second variables, respectively, and (L.O.T.) are lower-order terms and do not include forth derivatives of Z . The terms in (L.O.T.) can be absorbed into the left hand sides of (3.15) to (3.23). We have by integration by parts

$$\begin{aligned}
& -\int_0^\tau \int_0^\infty ([f_u] Z_{tXXX} + [f_v](W_1) Z_{tXX} + W_2 Z_{tXXX}) Z_{tXX} dx dt \\
&= -\int_0^\tau [[f_u] \frac{1}{2} Z_{tXX}^2]_0^\infty - \int_0^\infty [f_u]_x \frac{1}{2} Z_{tXX}^2 dx dt \\
&\quad - \int_0^\infty [[f_v] W_1 \frac{1}{2} Z_{tXX}^2]_0^\tau - \int_0^\tau ([f_v] W_1)_t \frac{1}{2} Z_{tXX}^2 dx dt \\
&\quad - \int_0^\tau [[f_v] W_2 \frac{1}{2} Z_{tXX}^2]_0^\infty - \int_0^\infty ([f_v] W_2)_x \frac{1}{2} Z_{tXX}^2 dx dt \\
&= \int_0^\tau ([f_u] \frac{1}{2} Z_{tXX}^2)(0, t) dt + \int_0^\tau \int_0^\infty [f_u]_x \frac{1}{2} Z_{tXX}^2 dx dt \\
&\quad - \int_0^\infty ([f_v] W_1 \frac{1}{2} Z_{tXX}^2)(\tau, x) dx + \int_0^\tau ([f_v] W_1)_t \frac{1}{2} Z_{tXX}^2(0, x) dx
\end{aligned}$$

$$\begin{aligned}
& - \int_0^\tau \int_0^\infty ([f_v] W_1)_t \frac{1}{2} Z_{txx}^2 dx dt + \int_0^\tau ([f_v] W_2)_t \frac{1}{2} Z_{txx}^2(0, t) dt \\
& + \int_0^\tau \int_0^\infty ([f_v] W_2)_x \frac{1}{2} Z_{txx}^2(0, x) dx dt.
\end{aligned}$$

Since $[f_u]$, $[f_v]$, W_1 and W_2 are functions of Z_t , Z_x , φ and ψ , the above expression involves only up to the third derivatives of Z . Similar treatment applies to other terms, and we conclude that the last integrals in (3.21) ~ (3.23) are

$$\begin{aligned}
& \varepsilon \int_0^\tau \int_0^\infty \sum_{j=1}^3 |\nabla^j Z|^2 dx dt + \varepsilon \int_0^\infty \sum_{j=0}^3 |\nabla^j Z|^2(x, 0) + |\nabla^j Z|^2(x, \tau) dx \\
& + \varepsilon \int_0^\tau \sum_{j=1}^3 |\nabla^j Z|^2(0, t) dt.
\end{aligned}$$

Using what we have acquired so far, we now would like to estimate the following quantity:

$$\begin{aligned}
& \alpha^4(3.15) + \alpha^3(3.16) + \alpha^2(3.17) + C_1(3.18) + \alpha(3.19) \\
& + \alpha(3.20) + (3.21) + (3.22) + C_2(3.23),
\end{aligned} \tag{3.30}$$

where α is a small positive constant to be fixed later and C_1 and C_2 are also positive constant to be fixed. Indeed, we will take as $\alpha \ll C_i$ where $i = 1$ or 2 .

When $\delta \ll 1$, (3.30) can be written as:

$$\begin{aligned}
& \alpha^4 \left\{ \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, \tau)|^2 dx + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + \sum_{j=1}^3 |\nabla^j|^2 dx dt \right\} \\
& \leq O(1) \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 dx + \underbrace{\varepsilon \int_0^\tau \sum_{j=0}^3 |\nabla^j Z(0, t)|^2 dt}_{B_0} \\
& + O(1) \underbrace{\left\{ \alpha^4 \int_0^\tau (\lambda_1 + \lambda_2)(0) Z_t Z(0, t) - \frac{|(\mu h_v)(0)|}{2} |Z(0, t)|^2 dt \right\}}_{B_1} \\
& + \alpha^3 \underbrace{\int_0^\tau \frac{(\lambda_1 + \lambda_2)(0)}{2} |Z_t(0, t)|^2 dt}_{B_2} \\
& + C_1 \underbrace{\int_0^\tau \frac{(\lambda_1 + \lambda_2)(0)}{2} |Z_{tt}(0, t)|^2 dt}_{B_4} + \underbrace{\alpha \int_0^\tau (\lambda_1 \lambda_2)(0) Z_{xxx} Z_{xx}(0, t) dt}_{B_6}
\end{aligned}$$

$$\begin{aligned}
& + \underbrace{\int_0^\tau (\lambda_1 \lambda_2)(0) Z_{xxx} Z_{xxt}(0, t) + \frac{(\lambda_1 + \lambda_2)(0)}{2} |Z_{txx}(0, t)|^2 dt}_{B_7} \\
& + C_2 \underbrace{\int_0^\tau \frac{(\lambda_1 + \lambda_2)(0)}{2} |Z_{ttt}(0, t)|^2 dt}_{B_9}. \tag{3.31}
\end{aligned}$$

We now need to estimate the boundary terms B_i , $i = \{1, 2, 4, 6, 7, 9\}$ which appears on the right hand side of (3.20 + i). Because ε is some positive constant which can be made small by taking δ small, we can choose ε and α to satisfy $\varepsilon \ll \alpha$. This allows us to treat some parts of B_i . For instance, $\varepsilon \int_0^\tau |Z_t(0, t)|^2 dx$ can then be absorbed in B_2 .

The crucial assumption

$$(\lambda_1 + \lambda_2)(0) < 0 \tag{3.32}$$

allows us to bound B_2 in (3.31). This in turn yields estimates for boundary terms B_i .

3.4 Evaluation of Boundary Terms

In this section, the assumption (3.32) is frequently used. Since $Z_x(0, t) = 0$, following terms in B_i ,

$$Z_x(0, t) = 0 \quad Z_{xt}(0, t) = 0 \quad Z_{xtt}(0, t) = 0$$

are zero. First, we treat the term $\int_0^\tau (\lambda_1 \lambda_2)(0) Z_{xxx} Z_{xxt}(0, t) dt$ in B_7 . Differentiate both sides of (3.14) with respect to t and x , and obtain

$$Z_{ttt} + (\lambda_1 + \lambda_2) Z_{xtt} + \lambda_1 \lambda_2 Z_{xxt} - h_v(Z_{tt} + \mu Z_{xt}) = R_t, \tag{3.33}$$

$$\begin{aligned}
& Z_{ttx} + (\lambda_1 + \lambda_2) Z_{txx} + \lambda_1 \lambda_2 Z_{xxx} - h_v(Z_{tx} + \mu Z_{xx}) \\
& + \underbrace{(\lambda_1 + \lambda_2)_x Z_{tx} + (\lambda_1 \lambda_2)_x Z_{xx} - h_{vx}(Z_{tt} + \mu Z_{xt}) - h_v \mu_x Z_{xt}}_{L(x, t)} = R_x. \tag{3.34}
\end{aligned}$$

Evaluate (3.33) and (3.34) at $(0, t)$:

$$Z_{ttt}(0, t) + (\lambda_1 \lambda_2)(0) Z_{xxt}(0, t) - h_v(0)(Z_{tt}(0, t) = R_t(0, t), \tag{3.35}$$

$$\begin{aligned}
& (\lambda_1 + \lambda_2)(0, t) Z_{txx}(0, t) + (\lambda_1 \lambda_2)(0, t) Z_{xxx}(0, t) \\
& - (h_v \mu)(0) Z_{xx}(0, t) + L(0, t) = R_x(0, t). \tag{3.36}
\end{aligned}$$

Integrate (3.36) $Z_{xxl}(0, t)$ over $0 \leq t \leq \tau$, we obtain

$$\begin{aligned}
 & \int_0^\tau (\lambda_1 \lambda_2)(0, t)(Z_{xxx} Z_{txx})(0, t) dt \\
 &= - \int_0^\tau (\lambda_1 + \lambda_2)(0, t) Z_{xxl}(0, t)^2 dt \\
 &+ \int_0^\tau (h_v \mu)(0)(Z_{xx} Z_{xxl})(0, t) + L(0, t) Z_{xxl}(0, t) dt \\
 &+ \int_0^\tau R_x(0, t) Z_{xxl}(0, t) dt. \tag{3.37}
 \end{aligned}$$

Using the smallness conditions (3.24), (3.27) and $\delta(\tau) \ll 1$, we obtain

$$\begin{aligned}
 & \int_0^\tau (\lambda_1 \lambda_2)(0, t)(Z_{xxx} Z_{txx})(0, t) dt \\
 & \leq O(1) \int_0^\tau Z_{xxl}(0, t)^2 dt + \varepsilon \int_0^\tau \sum_{j=1}^3 |\nabla^j Z|^2(0, t) dt. \tag{3.38}
 \end{aligned}$$

Now move all the terms on the right hand side to the left hand side of (3.35) except $Z_{xxl}(0, t)$, then square both sides:

$$\{(\lambda_1 \lambda_2)(0) Z_{xxl}(0, t)\}^2 = \{Z_{utl}(0, t) - h_v(0)(Z_{ut}(0, t) - R_t(0, t))\}^2. \tag{3.39}$$

Integrating (3.39) over $0 \leq t \leq \tau$, we obtain

$$\begin{aligned}
 & \int_0^\tau (Z_{xxl}(0, t))^2 dt \\
 & \leq O(1) \int_0^\tau Z_{utl}(0, t)^2 + Z_{ut}(0, t)^2 dt + \varepsilon \int_0^\tau \sum_{j=1}^3 |\nabla^j Z|_2(0, t) dt. \tag{3.40}
 \end{aligned}$$

Combine (3.38) and (3.40), we have

$$\begin{aligned}
 & \int_0^\tau (\lambda_1 \lambda_2)(0)(Z_{xxx} Z_{txx})(0, t) dt \\
 & \leq C_3 \int_0^\tau Z_{utl}(0, t)^2 + Z_{ut}(0, t)^2 dt + \varepsilon \int_0^\tau \sum_{j=1}^3 |\nabla^j Z|_2(0, t) dt. \tag{3.41}
 \end{aligned}$$

This bounds the term $\int_0^\tau (\lambda_1 \lambda_2)(0)(Z_{xxx} Z_{txx})(0, t) dt$ in B_7 . The term

$$\int_0^\tau (\lambda_1 \lambda_2)(0)(Z_{xxx} Z_{xx})(0, t) dt$$

in B_6 is treated similarly.

Since we can choose C_1 and C_2 large enough to satisfy $C_1 > C_3$ and $C_2 > C_3$, we can make right hand side of (3.31) be

$$\begin{aligned} O(1) \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 dx + \varepsilon \int_0^\tau \sum_{j=0}^3 |\nabla^j Z(0, t)|^2 dx \\ + O(1) \underbrace{\left\{ \int_0^\tau (\lambda_1 + \lambda_2)(0) Z_t Z(0, t) - \frac{|(\mu h_v)(0)|}{2} |Z(0, t)|^2 dt \right\}}_{B_1} \\ + \frac{(\lambda_1 + \lambda_2)}{2} (0) \int_0^\tau (Z_t^2 + Z_{tt}^2 + Z_{txx}^2 + Z_{utt}^2) dt \Big\}. \end{aligned}$$

Next we evaluate the first part of B_1 .

$$\int_0^\tau (\lambda_1 + \lambda_2)(0) Z_t Z(0, t) dt = \frac{\lambda_1 + \lambda_2}{2} (0) \{Z^2(0, \tau) - Z^2(0, 0)\}.$$

Consequently, from (3.38) and the discussion above we obtain

$$\begin{aligned} Z(0, \tau)^2 + \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, \tau)|^2 + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + \sum_{j=0}^3 |\nabla^j Z|^2 dx dt \\ \leq O(1) Z(0, 0)^2 + O(1) \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 \\ + \varepsilon \underbrace{\int_0^\tau \sum_{j=1}^3 |\nabla^j Z(0, t)|^2 dt}_{B_0} \\ + O(1) \int_0^\tau \frac{\lambda_1 + \lambda_2}{2} (0) (Z_t^2 + Z_{tt}^2 + Z_{txx}^2 + Z_{utt}^2)(0, t) \\ - \frac{|(\mu h_v)(0)|}{2} |Z(0, t)|^2 (0, t) dt. \end{aligned} \tag{3.42}$$

Integrations of Z_t^2 , Z_{tt}^2 , Z_{txx}^2 , and Z_{utt}^2 in B_0 can be absorbed in the last integrals in (3.42). Note that $(\lambda_1 + \lambda_2)(0) < 0$ and $\varepsilon \ll \alpha$.

It remains to evaluate

$$\varepsilon \int_0^\tau Z_{xx}(0, t)^2 dt \quad \text{and} \quad \varepsilon \int_0^\tau Z_{xxx}(0, t)^2 dt.$$

We start with the second one. For this we move all the terms on the left hand side of (3.36) to the right hand side except Z_{xxx} , then square both sides, and integrate over $0 \leq t \leq \tau$:

$$\begin{aligned} \int_0^\tau Z_{xxx}(0, t)^2 dt &\leq O(1) \int_0^\tau \{ -(\lambda_1 + \lambda_2)(0) Z_{txx}(0, t) - (h_v \mu)(0) Z_{xx}(0, t) \\ &\quad + L(0, t) + R_x(0, t) \}^2 dt \\ &\leq O(1) \int_0^\tau Z_{txx}(0, t)^2 + Z_{xx}(0, t)^2 dt \\ &\quad + \varepsilon \int_0^\tau \sum_{j=0}^3 |\nabla^j Z(0, t)|^2 dt. \end{aligned}$$

Hence, $\int_0^\tau Z_{xxx}(0, t)^2 dt$ is expressed by other terms in B_0 .

Next we evaluate $\int_0^\tau Z_{xx}(0, t)^2 dt$. We have from Sobolev's inequality that:

$$\begin{aligned} \int_0^\tau Z_{xx}(0, t)^2 dt &\leq \int_0^\tau \left\{ \sup_x |Z_{xx}(0, t)| \right\}^2 dt \\ &\leq \int_0^\tau \int_0^\infty Z_{xxx}(x, t)^2 + Z_{xx}(x, t)^2 dx dt. \end{aligned}$$

Consequently, $\varepsilon \int_0^\tau Z_{xx}(0, t)^2 dt$ can be absorbed in the right hand side of (3.42) because $\varepsilon \ll \alpha^4$. Then neglecting the last term of (3.42), we obtain the following estimate:

$$\begin{aligned} Z(0, \tau)^2 + \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, \tau)|^2 + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + \sum_{j=1}^3 |\nabla^j Z|^2 dx dt \\ \leq O(1) \left\{ Z(0, 0)^2 + \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 \right\} dx \end{aligned} \quad (3.43)$$

$$\leq O(1) \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, 0)|^2 dx. \quad (3.44)$$

The last inequality comes from a simple application of the Sobolev's lemma:

$$Z(0, 0)^2 \leq \sup_x Z(x, 0)^2 \leq C \int_0^\infty Z(x, 0)_x^2 + Z(x, 0)^2 dx.$$

The right hand side of (3.43) also contains the derivatives of Z with respect to t which are not a part of the initial data. Nevertheless, using (3.9), the derivatives of Z with respect to t can be converted to Z , w , and their derivatives with respect to x . The estimate for w can be derived from (3.11), (3.13), and (3.43). In conclusion, we have the desired energy estimate:

$$\begin{aligned} Z(0, \tau)^2 + \int_0^\tau \sum_{j=0}^3 \left| \frac{\partial^j}{\partial x^j} Z(x, \tau) \right|^2 + \sum_{j=0}^2 \left| \frac{\partial^j}{\partial x^j} w(x, \tau) \right|^2 dx \\ + \int_0^\tau \int_0^\infty |\mu_x| Z^2 + \sum_{j=1}^3 |\nabla^j Z|^2 + \sum_{j=1}^2 |\nabla^j w|^2 dx dt \\ \leq O(1) \int_0^\infty \sum_{j=0}^3 \left| \frac{\partial^j}{\partial x^j} Z(x, 0) \right|^2 + \sum_{j=0}^2 \left| \frac{\partial^j}{\partial x^j} w(x, 0) \right|^2 dx \end{aligned} \quad (3.45)$$

provided that $\delta(\tau) \ll 1$.

This completes the proof of Lemma 3.1.

THEOREM 3.2. *(u_0, v_0) is some point on the wave connecting (u_+, v_+) and (u_-, v_-) . Also $(\lambda_1 + \lambda_2)/2 < \sigma$ is satisfied at (u_0, v_0) . Suppose that $(\varphi, \psi)(x - \sigma t)$ is a smooth traveling wave of (1.1) satisfying $\lambda_1 < \sigma < \lambda_2$, (3.24), (3.25) and (3.26) and that it is connecting (u_+, v_+) and (u_0, v_0) . Then any perturbation (3.1) of (φ, ψ) give a rise to a global solution (u, v) of (1.1) which tends to $(\varphi, \psi)(x - \sigma t)$ uniformly in x as $t \rightarrow \infty$, provided that $\delta(0)$ is sufficiently small.*

Proof. It is enough to show the case $\sigma = 0$ only. That the solution exists globally in time follows from the a-priori estimate (Lemma 3.1) and the local existence theory (see Remark 3.2). The convergence of (u, v) to (φ, ψ) is equivalent to $(Z_x, w) \rightarrow 0$ as $t \rightarrow \infty$. Since (3.45) holds for $\tau = \infty$, we have

$$\begin{aligned} Z_x(x, t)^2 + w(x, t)^2 &= - \int_x^\infty (2Z_x Z_{xx} + 2w w_x)(y, t) dy \\ &\leq 2 \int_x^\infty (|Z_x Z_{xx}| + |w w_x|)(y, t) dy \\ &\leq C \int_x^\infty (Z_x^2 + w^2)(x, t) dx \\ &\quad \times \int_x^\infty (Z_{xx}^2 + w_x^2)(x, t) dx \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This completes the proof of the theorem. ■

Remark 3.1. We may alternatively define

$$\delta(\tau) \stackrel{\text{def}}{=} |u_+ - u_0| + |u_+ - \hat{u}| \\ + \max_{0 \leq t \leq \tau} \int_0^\infty \sum_{j=0}^3 |\nabla^j Z(x, t)|^2 + \sum_{j=0}^2 |\nabla^j w(x, t)|^2 dx$$

in Theorem 3.2 and Lemma 3.1, where \hat{u} is defined to be a number, uniquely determined satisfying $\lambda_*(\hat{u}) = \sigma$, since under this definition of δ , the conditions (3.24), (3.25) and (3.27) are satisfied when δ is small.

Remark 3.2. The following theorem is needed in the proof above.

LOCAL EXISTENCE THEOREM. *Suppose that the initial data $(u, v)(x, 0)$ is in C^1 , then there exists some small $\tau > 0$ such that in domain $D(\tau) = \{(x, t); 0 \leq t \leq \tau, \sigma t \leq x\}$, the system (1.1), (3.1) and (3.2) has a unique C^1 solution. Here, δ depends only on supremum norms of the initial data and its first derivatives.*

The proof is given by the standard iteration method, (see [4] and [5]). If we assume that the differentiability and the compatibility of initial and boundary data up to third derivatives, our local solution becomes C^3 and Z becomes C^4 . Even if those conditions fail, we should still be able to prove Lemma 3.1 and Theorem 3.2 by using Friedrich's mollifier, though there might be some discontinuities in the solution thus obtained.

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